Afterword

Since the fourth and last edition of *Number* was published, a half-century ago, mathematics has advanced with astonishing speed. Several of the most outstanding unsolved problems have either been solved or spread roots to new places in nearby fields. From the time our ancestors first discovered rules for operating with numbers, problems of mathematics cropped up; some were solved, others not; but, like stones in ancient Phoenician barley fields, new ones surfaced faster than the old were removed. Yet, despite developments in modern number theory and analysis, the content of *Number* is still as fresh as when the first edition was published in 1930. Reading *Number* today, the mathematics enthusiast is struck by its lucid language, contemporary relevance, and intellectual provocation.

Progress in mathematics has accelerated. On the surface, it may seem as if only a few famous problems have been solved in the last fifty years. But modern mathematics has increasingly become more profound. Solutions to surface problems—the so-called "gems"—are inextricably linked to others that are often fields apart, crossing boundaries by intricately tangled roots coming from one great and stable unifying source.

The ancient problems of doubling of the cube, trisecting the angle, and squaring the circle remained a mystery for two thousand years, waiting for the brilliant ideas of modern algebra to uncover their proofs. In 1837 Pierre Wantzel proved that it is impossible to duplicate the cube or trisect an arbitrary angle, thereby solving the two great mysteries of antiquity. Was that the end of the long story that began with the tale of the oracle at Delos, which claimed that relief of the devastating plague in Athens would come when the cubic altar to Apollo would be doubled in size? Certainly not!

Wantzel's solution opened new questions, questions on which simple algebraic criteria would permit geometric constructions as solutions of rational polynomial equations. These questions, in turn, opened the far broader question of how to convert geometry to the theory of equations.

Dantzig focused on the evolution of the number concept to keep his book well within a manageable scope, staying reasonably clear of the more geometric branches of mathematics, even though he knew that answers to some of the most elementary questions of number theory are sometimes best handled through sophisticated geometry. His book mentions the Goldbach Conjecture, the Twin Prime Conjecture, Fermat's Last Theorem; three of many outstanding statements, still unproven at the time of its last printing. Fermat's Last Theorem was solved late in 1994, but the other two conjectures remain unsolved.

The Twin Prime Conjecture, for example, is one of a large assortment of problems prompted by asking simple, phenomenological questions about how the collection of prime numbers is distributed among all natural numbers. The wonderful thing about many of the finest questions in number theory is that they can be stated so simply. They require little or no technical language to understand and can often attract the least suspecting visitor, who—if not careful—may find him- or herself absorbed in endless hours of mathematical diversions. How many prime numbers are there of the form n^2+1? How many prime numbers p are there with $2p + 1$ being a prime number? Are there any odd perfect numbers? (Perfect numbers, like 6, are equal to the sum of their own divisors.) We now know that there are none under 300 digits. But are there any? We know that if one exists at all it must be a sum of squares and at the same time have at least forty-seven prime factors. But are there any at all?

There was a time when young, naïve mathematicians (like myself) would worry about what would happen when all these fine questions—those simply stated ones—would be solved. We have learned not to worry. Not only will there always be enough fine questions to tempt the dilettante, but each answer will breed a family of new ones. Such was certainly the case with Fermat's Last Theorem, which reared much of modern number theory; it was also the case with those stubborn Ancient Greek problems, which formed so much of modern algebra. We forever find ourselves at the relatively earlier stages of understanding number.

Fifty years may seem like a long time to wait for solutions to outstanding problems, but considering that some have waited millennia it seems that plenty has happened in the mere 2 percent of the time since Euclid's *Elements* first appeared and modern mathematics took off. First, we'll look at how computers have affected mathematics. Then we'll take a peek at the progress on the Goldbach Conjecture, Fermat's Last Theorem, and the Twin Prime Conjecture.

Computers

In 1954, the year the fourth edition of *Number* was published, MANIAC I (Mathematical Analyzer, Numerical Integrator and Computer) was the most advanced computer of the time, using 18,000 vacuum tubes. (One can only imagine how often the machine broke down because a single one of the eighteen thousand tubes failed.) In 1951, without the use of computers, the forty-four-digit number (2148+1)/17 =

2098893665744058648615126425661022259386392 1 was discovered as the largest prime, but just three years later, with the help of MANIAC I, the largest prime was discovered to be $2^{2,281} - 1$, a number with 687 digits. Today we know that $2^{24,036,583} - 1$ is a prime number. It contains 7,235,733 digits.

In 1954, graphics interface analogue printers were still on the drawing boards, though prototypes which moved styluses up, down, right or left according to the coordinates of input were being built by IBM. Dantzig does not mention the Riemann-Zeta function, but the zeros of that interesting function (solutions to the equation z(s) = 0) have a curious connection with the distribution of prime numbers. A flood of number theory theorems would automatically follow from a proof of the Riemann Hypothesis, which claims that all the zeros of z(s) are complex numbers of the form 1/2 + *ai*. For one, in 1962 Wang Yuan showed that if the Riemann Hypothesis is true, then there are infinitely many primes *p* such that *p* and *p* + 2 are a product of at most three primes. Riemann was able to compute the first three zeros of the zeta function with astonishing accuracy by hand. In 1954, when Alan Turing found 1,054 zeros of the zeta function without an electronic computer, 1,054 seemed like a huge number of zeros, but now, with the aid of modern computers, we know more than 10^{22} zeros and all of them are on the line having its real part equal to 1/2. Today, the world's fastest computer cannot possibly tell if *all* zeros of the Riemann-Zeta function lie on the vertical line 1/2 + *ai* in the complex plane, but a simple $500 desktop computer can instantly find many that do and none that do not.

But computers work with finite numbers and though they can work at astonishing speeds, those speeds are only finite. They can help discovery, relieve the mathematician of grueling endless computations and—in many cases—suggest possibilities that could never have been spotted by human reckoning.

Fermat's Last Theorem

Fermat's Last Theorem is true. This 350-year-old problem was solved in 1994 by Andrew Wiles, with the help of his former student Richard Taylor, using some of the most beautiful and brilliant ideas in number theory that recognize relationships between outwardly different mathematical objects coming from remotely different branches of mathematics.

We cannot presume to give anything near an adequate story here. At most, such a story can give only a highly brief account, naming very few of the many players who hammered out impressive ideas, and skipping exciting new dreams and major advances of the trip, saying nothing about the flying sparks coming from strikes, the brilliant inroads leading to those strikes, or the countless fires set by those sparks. The formal proof is highly technical, but it has been comprehensively outlined in several popular books listed in the Further Readings section.

The Goldbach Conjecture

We now know a bit more about the Goldbach conjecture, which says that every even number greater than 2 can be written as a sum of two primes. Dantzig knew, though didn't mention, that every sufficiently large odd number can be written as a sum of three primes. The Russian mathematician Ivan Vinogradov proved this in 1937. Dantzig also knew the wild but interesting theorem that claimed that every positive integer could be written as the sum of not more than 300,000 primes. Now that may seem like a long way off from Goldbach's conjecture, but in fact 300,000 is a lot less than infinity! Lev Shnirelmann, another Russian, proved it in 1931. Soon after, Vinogradov used methods of Hardy, Littlewood and Ramanujan to prove that any sufficiently large number could be written as a sum of four primes. In more precise terms, it means that there exists some number *N* such that any integer greater than *N* can be written as a sum of four primes. This brought down the number of primes in the sum at the expense of the size of the number for which the conjecture would be true.

Vinogradov proved both theorems by exhibiting a contradiction from the assumption that infinitely many integers cannot be written as a sum of four primes. His proof could not specify how large *N* had to be, but in 1956, K.G. Borodzkin showed that *N* had only to be greater than $10^{4,00,8,660}$, a number with more than four million digits. It is now known that "almost all" even numbers can be written as the sum of two primes. "Almost all" here means that the percentage of even numbers under *N* for which the Goldbach conjectures are true tends toward 100 as *N* grows large. Just after the last printing of *Number*, there was a flurry of theorems closing in on the classical Goldbach Conjecture. First, it was proven that every sufficiently large even integer is the sum of a prime and a product of at most nine primes. As the years went by, the product was reduced, first to 5, then to 4, then to 3 and finally to 2. We now know that every sufficiently large even integer is the sum of a prime and the product of two primes. We also now know that one Goldbach variation is true: with a finite number of exceptions, every even number is a sum of a pair of twin primes.

Twin Primes

It is still not known whether or not there are an infinite number of twin primes, but it seems certain that there are. Perhaps the answer is beyond the current resources of mathematics. But there is another, stronger twin prime conjecture which states that the number of twin primes less than *x* grows close to another fully calculable number that depends on *x*. Clearly, this strong twin prime conjecture implies the usual twin prime conjecture. The first few pairs of twin primes are (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73), (101,103). Today, the largest known twin primes have more than 24,000 digits. It is interesting to note that in 1995 T. R. Nicely used the twin primes 824,633,702,441 and 824,633,702,443 to discover a flaw in the IntelPentiummicroprocessor.

As with the Goldbach Conjecture, after the last edition of *Number* was published, a flood of theorems converged toward the twin prime conjecture. Since 1919 we knew that there are infinitely many numbers *k* such that both *k* and *k* + 2 are products of at most nine primes. Just after the last edition of *Number*, it was discovered that *k* and *k* + 2 are products of at most three primes.

Computer programmers building tests, giving machines heated workouts, are hitting many of these conjectures, optimistically searching for more twin primes or zeros of the zeta function. Why do they bother? No matter how many twin primes or zeros they find, they could never prove the conjectures that way. They are not trying to prove anything, but rather trying to display what theorists believe exists. Each new find contributes to confidence in the conjecture. Pessimists would hope to find a zero of the zeta function off the magic line to give a counterexample. That’s possible. But if the first 1022 zeros follow Riemann's prediction, how likely would it be that the next will not? And then we must ask this question: Riemann checked only the first three zeros, so how could he have possibly known that they would all lie on the line with real part equal to 1/2? Answer: He knew something about the character, purpose and destination of the whole beast, not just what it is when it stops to pick up another zero.

This limited selection is a sampling of some of the countless jewels of mathematics that were solved, advanced or remained too tough to crack in the past fifty years. The choices here are limited to the subjects treated in *Number* and hence more connected to the field of number theory. However, readers of *Number* should be aware that though few of the prize problems mentioned in *Number* have been solved, the past fifty years of attempts at solving problems like them have given us a higher—much higher—comprehension of the things we do when we do mathematics. We now see it all coming from that one great and stable unifying source—the *thing* that *is* mathematics. This viewpoint was unavailable to Dantzig and other mathematicians working in the first half of the twentieth century.

We know also—just as Dantzig did back in 1954—that great theorems of mathematics tidily unveil themselves in one branch to cast teasing silhouettes on delicate curtains separating others. Perhaps some curtains will gently separate in the breeze of the next fifty years.